# LOGIC PRIMER WORKBOOK

## 13 APPENDIX C: SYMBOLIC LOGIC

| A. Introduction | D. Truth Table Methods | G. Formal Proof |
|---|---|---|
| B. Sentential Calculus | E. Shorthand Technique | H. Conditional Proof |
| C. Interdefinability | F. Valid Inference Rules | I. Indirect Proof Method |

## A. INTRODUCTION

It is now generally recognized that symbolic logic or mathematical logic represents at the least a development of techniques and concepts that were implicit in classical logic. Hence, it is not surprising that much of the symbolism and operations is clearly correlated with the material already presented in the Logic Primer. Accordingly, at the risk of some repetition, certain considerations warrant a restatement for effecting an orderly transition from traditional to modern formulation.

### Arguments

An *argument* is construed as a series of claims such that at least one of such claims declares a *condition* under which at least one other of such claims declares a *conclusion*. The condition claim is called a *premise*. In short, an argument consists of a series of propositions in support of another proposition – the conclusion.

Arguments are made explicit by expressing them as declarative sentences related to one another by negation, conjunction, disjunction, implication, or equivalence, and by the use of punctuational devices. Some arguments, although their character does not require the implicative relation, are conventionally formulated as implicative, using the "if ... then" format. The premise sentence is placed between the "if" and the "then," whereas the conclusion sentence is placed after the "then." The premise as it is expressed in such a formula is called the antecedent; the conclusion as it is expressed in such a formula is called the consequent.

In explicit arguments, *truth* is construed as a matter of accurate correlation or application claimed between the sentences making up the arguments and whatever those sentences are about. *Validity*, on the other hand, is construed as concerned with the relations claimed as obtaining among those sentences making up an argument. One consequence of this consideration is that strictly speaking truth and falsity are treated as semantical (applicational) rather than logical (relational) concepts whereas validity and invalidity are logical (relational) rather than semantical (applicational) concepts. Logic in use must satisfy both logical and semantical requirements. Such satisfaction is sometimes indicated by calling an argument *sound*. In this way, logical ordering and semantic application combine

to produce sound arguments; and this outcome is what interests us most from a practical point of view. The universe is ordered in conformity with logical relations; it remains a presupposition to treat our thought and language that way. Human beings do express themselves more or less clearly in the manner presumed by our logical operations.

Implementation of explicit arguments in terms of truth yields the distinction between *true declarative sentences* and *false declarative sentences*; likewise, in terms of validity, arguments are said to be valid or invalid. Again, by convention, "valid" and "invalid" are labels commonly reserved for the implicative formulation of arguments. An argument which is valid is not invalid; an argument which is invalid is not valid. Truth-functional validity requires that in no application (or in no instance) is an argument's consequent false and its antecedent true. A sound argument requires that in every application (or in every instance) the antecedent being true the consequent of that antecedent must be true. Should a sound argument's claim be conjoined with the claim that the consequent of the sound argument being true the antecedent of that argument must be true, this conjunction of claims is called an equivalence, expressed in the "if and only if" format. It is important to note that truth-functional formulations of arguments make explicit what is required to satisfy the claim made. Such formulations do not provide us with the means to satisfy the requirement. The actual universe is ordered in accordance with our logical correlations, including the language used in the statements.

There are three important characteristics of symbolic logic that we shall cite and seek to keep in mind as we examine statements and their relations.

- o The use of ideograms or signs to stand directly for concepts. For example, the subtraction sign ("-") or the multiplication sign ("x").

- o The use of deductive method, which has as one of its important features the following: from a small number of statements, we can generate a number of other statements by the application of a number of rules.

- o The use of variables, these having a definite range of significance, as we shall indicate.

For example, consider the following:

1. If Leo is broke then a new car is out of the question. But, a new car is not out of the question. Therefore, Leo is not broke.
2. If this specimen is an insect, it will have only three pairs of legs. You can see that it does not have three pairs of legs. Therefore, it must not be an insect.

Examination of these arguments reveals that there are prominent resemblances that can be made explicit if we substitute for the constituent statements of the two arguments the letters "p" and "q." Substitution results in:

3. If p, then q. But not-q. Therefore, not-p.

The use of symbols, letters of the alphabet in this case, functioning as variables enables logicians to make clear the structure of arguments. The use of symbolized variables also permits logicians to formulate general rules for testing the validity of arguments and enables them to classify arguments into types. Finally, the use of symbols enables the logician to express complicated statements in a

concise and economical manner and to express concepts that are technically useful for symbolic logical operations.

Logical Forms

Logicians are interested in *relations* among *variables*. This is commonly referred to as an interest in *logical form*. Some of the most prominent relations traditionally occupying the attention of logicians are: negation, expressed as "not"; disjunction, expressed as "or"; conjunction, expressed as "and"; implication expressed as "if - then ..." and equivalence, expressed as "if and only if." Variables of interest are ambiguous indicators of whatever data are to be related, for example words, sentences, relations, and the like. One common mode of expressing such variables is by means of the lowercase alphabet letters, such as "p," "q," "r," and so on. Special uses are made of certain parts of the alphabet.

In sum, logical argument consists of logical form or logical formulation of relations among variables; such logical form is symbolized by means of conventional notation. Although argument forms can in principle be expressed as claims in declarative form, a useful subtype of such argument forms is the implicative form mentioned above. Henceforth, we will preserve this distinction by reserving the label *declarative* for those forms other than the implicative. One of the logicians' interests has been that of developing methods for testing the validity of the implicative argument form.

# B. THE SENTENTIAL CALCULUS

Sentential calculus (sometimes called the *propositional calculus*) plays a fundamental role in symbolic logic. Generally, logicians take all sentences that are either true or false but not both as the province to which this calculus applies. For example, the following are sentences of the declarative type:

1. Plato was not a musician.

2. Plato was a philosopher, or he was not a student of Socrates.

3. Plato was a philosopher, and he knew Socrates.

The following are sentences of the implicative type:

4. If Plato was a philosopher, then he knew Socrates.

5. Plato was a student of Socrates only if Socrates was Plato's teacher.

The following is a sentence of equivalence type that consists of two sentences of implicative type conjoined to form a sentence of declarative type.

6. Plato was a student of Socrates if and only if Socrates was Plato's teacher.

These and other sentences, compounded and negated to various degrees of complexity, constitute the subject matter to which the sentential calculus of symbolic logic is applied.

Sentences of the types illustrated above are said to have a truth-functional formulation in that the truth-value of the sentence (its being true or false but not both) as formulated is uniquely determined by the truth-value of the simplest declarative parts of the formulation. While, according to the convention adopted, "valid or invalid" (but not both) rather than "true or false" (but not both) are

applicable to the illustrative sentences at 4 and 5 above, the validity status of sentences of such type is also uniquely determined by the truth status of the antecedent declarative part and the truth status of the consequent declarative part.

To illustrate such truth-functional formulation, sentence 3 above can be symbolized as "p and q." As a condition for the truth of "p and q," one which is not contrary to ordinary-language use, we stipulate that "p and q" is true only under one set of conditions, namely when both the sentence for which "p" is substituted and the sentence for which "q" is substituted are true; under any other truth-value condition "p and q" will be false. Such dependence of truth-value can be represented by a truth table.

| Plato was a philosopher. | He knew Socrates. | Plato was a philosopher, and he knew Socrates. |
|---|---|---|
| p | q | p and q |
| true | true | true |
| true | false | false |
| false | true | false |
| false | false | false |

Any statement can be represented in a truth table by making it part of an argument in which the statement and its denial are treated as two propositions. By taking "Plato was a musician" as the simplest declarative part of the sentence at 1, above, its denial can be formulated as "not-p"; the truth table for this formulation would be:

| Plato was a musician. | Plato was not a musician. |
|---|---|
| p | not-p |
| true | false |
| false | true |

Before constructing truth tables to represent other types of sentences, it will be convenient to symbolize the relations as well as the sentences related.

By a common convention, sentences will be symbolized by means of the middle lowercase letters of the alphabet, that is, "p," "q," "r," "s," up to but not including "v," "w," "x," "y," and "z." Lowercase letters occurring in the alphabet prior to "p" and after "u" are often used for other logic purposes not covered here. The letters "p" through "u" are referred to as sentential variables.

The relations expressed by the words "not," "and," "or," "if … then …," "if and only if" and "therefore" are symbolized as follows:

| " ~ " | for | "not" |
|---|---|---|
| " . " | for | "and" |
| " v " | for | " or "* |
| " ⊃ " | for | "if then…" |
| " ≡ " | for | "if and only if" |
| " ∴ " | for | "therefore" |

(*Used in the sense of "at least one, possibly but not necessarily both" of two variables. The "or" is used in this sense; although "or" has a more restrictive sense of "at least one but not both" or 'either but not both".)

Such symbols are referred to as logical connectives. As a consequence of full symbolization, the sentence, for example, "If Plato was not a musician then Plato was a philosopher" would be notated as "~p ⊃ q". It is to be noted that such symbolization makes explicit the logical form of the sentence without restricting the logical form to that particular sentence. Obviously, there is an indeterminate number of sentences occurring in the same logical form only one of which was selected here for illustrative purposes. Another convenience we will adopt will be to notate "true" as "T," "false" as "F." Thus, "T" and "F" indicate assumed truth-values.

Before we consider some of the basic truth tables of the sentential calculus, it is necessary to provide symbolic notation for logical punctuation in order to indicate the scope of the logical connectives. Parentheses, brackets, and braces are used in symbolic notation to avoid ambiguity. For example, the following would be quite ambiguous without punctuation:

~ p · q v r ⊃ s ≡ ~ t

However, we avoid ambiguity by clustering or gathering the variables in a certain way; for example, one such case:

~ [(p · q) v (r ⊃ s)] ≡ ~ t

Of course, these variables and logical connectives could be punctuated differently. Accordingly, there is need for a couple of rules to guide punctuational procedure:

Rule I: The "~" applies to the largest punctuational segment immediately to the right. This punctuational segment may be a single variable as in "~ p", a complex within parentheses such as in "~ (p · q)," a complex within brackets as in "~ [(p · q) v r]," or a complex within braces as in "~ {[(p · q) v r] ⊃ s}."

Rule II: Logical connectives other than "~" apply to the largest punctuational segments immediately flanking such connectives. Either of such segments may be a simple variable or a complex of whatever

internal makeup.

Consider the following simple cases:

1. ~ p · q

2. ~ (p · q)

3. (p v q) ≡ (r · s)

4. p · [(q ⊃ r) v (s ⊃ t)]

In 1, the " ~ " extends to "p".

In 2, the scope of " ~ " extends to the entire complex segment enclosed by the parentheses, that is, to "(p · q)".

In 3, the scope of " v" and " · " extends to the variable segments that flank them, that is to " p" and " q " and to " r " and " s ", respectively. The scope of " ≡ " extends to the segments which flank it, that is to "(p v q)" and "(r · s)". In 3, the logical connective with the largest scope is " ≡ ".

In 4, the " · " is the logical connective with the largest scope. The scope of " v " extends over the complex segments that flank it. The " ⊃ " extends to the variables that flank it.

We are now in a position to consider some basic truth tables of the sentential calculus. Horizontal sequences of "T's" and "F's" beneath the variables will be referred to as rows; vertical sequences will be referred to as columns.

The Contradictory Function:

| p | ~ p |
|---|-----|
| T | F   |
| F | T   |

The Conjunctive Function:

| p | q | p · q |
|---|---|-------|
| T | T | T     |
| T | F | F     |
| F | T | F     |
| F | F | F     |

The Disjunctive Function:

| p | q | p v q |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

The inclusive sense of "or" is represented above. The "(p v q)" in this sense means at least one, not requiring but permitting both to be true. The exclusive sense can be treated as a special case of the inclusive sense of "or", i.e., the sense wherein it is not the case that both are true.

The Implicative Function:

| p | q | p ⊃ q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

It is not difficult to find idiomatic uses of implicative statements that correspond to the first and last rows. Nevertheless, notwithstanding examples showing a correspondence between idiomatic uses of implicatives and the truth-functional representation of the "if-then" relations above, there are some difficulties with the truth-table interpretation for the logical connective " ⊃ ". These "difficulties" are usually referred to in the literature as the "paradox of material implication." This "paradox" can be expressed in two ways:

i.    The truth table for the implicative function shows that any true statement is implied by any true statement as well as any false statement (first and third rows).

ii.    The truth table further shows that any false statement implies any true statement as well as any false statement (third and fourth rows).

These "paradoxes" are avoided by clarity of claims made. The "paradox of material implication" is dispelled by the clear understanding that the "if-then" (the " ⊃ ") relation, as some logicians use it, is

not intended to restrict what can be meant, but solely to restrict truth-functional relations among whatever is meant by the data related. While any factual meaning that is shared by the statements related is not ruled out by the " ⊃ ", such factual meaning is not required for the " ⊃ " relation to function as a logical relation between statements that are either true or false. Logical implication in this sense is frequently distinguished from material implication, in that the former holds for a meaning relation dependent in part upon the factual status of the sentences related, whereas the latter holds for a truth-functional relation not dependent upon the factual status of the sentences related. Since we are concerned essentially with the truth-functional operation of " ⊃ ", logical implication will be interpreted in terms of a material-implication operation. Again, whatever we do mean by our sentences, we will develop a truth-functional means of ordering our sentences about that meaning.*

Accordingly, we adopt (assume) the terminology of "logical implication" (and "logical equivalence") in the sense of material implication (and material equivalence). Validity, truth status, and meaning, while related, are distinct; to ignore the distinction is to confuse the three and the result is a "paradox."(See References)

As an aid to symbolization correlated with implicative relations as they occur in ordinary language, some ordinary-language cues or indicator words are helpful:

| if p then q | $p \supset q$ |
| --- | --- |
| p only if q | $p \supset q$ |
| p thus q | $p \supset q$ |
| p therefore q | $p \supset q$ |
| p hence q | $p \supset q$ |
| p unless q | $\sim p \supset q$ |
| p if q | $q \supset p$ |
| p since q | $q \supset p$ |
| p because q | $q \supset p$ |
| p for q | $q \supset p$ |

The Equivalence Function:

| p | q | p ≡ q | ( p ⊃ q) · ( q ⊃ p) |
|---|---|-------|---------------------|
| T | T | T | T |
| T | F | F | F |
| F | T | F | F |
| F | F | T | T |

From the above, two things are clear:

(i) " p ≡ q " is true when both "p" and "q" have the same truth value (first and last rows), and

(ii) "p ≡ q " and " (p ⊃ q) · (q ⊃ p) " have identical truth tables.

Accordingly, "p ≡ q " is logically equivalent to " (p ⊃ q) · (q ⊃ p) ". These can be interchanged; i.e., one can be substituted for the other. (The importance of this kind of substitution will become clear in the section dealing with formal proofs.)

Finally, " ≡ " is sometimes called a "triple bar" or a "double horseshoe" and is an interpretation of the phrase "if and only if."

## C. INTERDEFINABILITY (RELATIONS BETWEEN TRUTH FUNCTIONS)

In the last section, it was shown that "p ≡ q" and "(p ⊃ q) · (q ⊃ p)" have identical truth tables. This identity is possible since the " ≡ " can be defined in terms of two other logical connectives, the " ⊃ " and " · ". It is possible also to define the " ⊃ " in terms of the " ~ " and the " · ". The " ~ " is taken as a primitive notation, that is, as one which is not defined in terms of other logical connectives. What follows shows interdefinability for the " ⊃ " in terms of the " ~ " and " · ", and in terms of the " ~ " and " v ". (Top row displays the sentential variables and logical expressions; the Bottom row consists of column number for reference.)

| p | q | p ⊃ q | ~(p · ~q) | ~p v q |
|---|---|-------|-----------|--------|
| T | T | T | T | T |
| T | F | F | F | F |
| F | T | T | T | T |
| F | F | T | T | T |

| (i) | (ii) | (iii) | (iv) | (v) |
|-----|------|-------|------|-----|

Columns (iii), (iv), and (v) are identical. Consequently, the complex truth functions are said to be logically equivalent and, therefore, substitutable one for another. Logically equivalent truth functions connected by the " ≡ " yield a column of "T"s" under the " ≡ " in a truth table.

## D. TRUTH TABLE METHOD TESTING VALIDITY OF ARGUMENTS

We are now in a position to construct and show the application of truth tables in testing the validity of arguments. It is important to keep in mind that in a particular argument the value of a variable remains constant. Further, the expression for which a particular variable has been substituted remains constant throughout all rows of a truth table.

Consider the following argument:

1. If Andy is a bachelor, then he must read Plato.

2. But he does not read Plato.

Therefore, Andy is not a bachelor.

Formulation of this argument as implicative using the "if-then ..." format yields:

$[(p \supset q) \cdot \sim q] \supset \sim p$

This truth function has two variables, namely, " p " and " q ". The number of rows in the truth table will be four. A handy formula for determining the number of rows in constructing any truth table is $R = 2^n$, where "R" is the number of rows and "n" is the number of different variables in the truth function.

|   | p | q | $[(p \supset q)$ | · | $\sim q]$ | $\supset$ | $\sim p$ |
|---|---|---|------|---|-----|-----|-----|
| 1 | T | T | T | F | F | T | F |
| 2 | T | F | F | F | T | T | F |
| 3 | F | T | T | F | F | T | T |
| 4 | F | F | T | T | T | T | T |
|   | (i) | (ii) | (iii) | (iv) | (v) | (vi) | (vii) |

From columns (i) and (ii), we obtain (iii), recalling the truth-functional formulation for the " ⊃ " (Section B, the Sentential Calculus).

From (ii), we calculate the truth-values in column (v) based on the truth-functional formation for the " ~ ".

From (iii) and (v), we obtain the truth-values in column (iv), similarly based on the truth-functional formulation for the " · ".

From (i), we determine the truth-values in (vii). Finally, from (iv) and (vii), we calculate the truth-values under the " ⊃ " in column (vi); this connective has the largest scope.

A truth function which yields all truth values as "true" under the connective with the largest scope (as is the case in the example above) is recognized as an always-true truth function. An always-true truth function is called a tautology. If a tautology is correlated with an implicative argument, then the argument is valid as are all other arguments having the same logical form.

A truth function which yields some truth values as "true" and others as "false" under the logical connective with the largest scope in a truth table is known as a contingent truth function. If a contingent truth function is correlated with implicative argument, that argument and all others, having the same logical form are recognized as invalid.

To illustrate, consider the following:

1. If Leo is a bachelor, then he reads Plato.

2. But everyone knows that Leo is not a bachelor.

Therefore, he does not read Plato.

Truth-functional formulation of the above argument yields:

$[(p \supset q) \cdot \sim p] \supset \sim q$

Constructing a truth table, we note that it will consist of four rows.

|  | p | q | [( p ⊃ q ) | · | ~ p | ⊃ | ~ q |
|---|---|---|---|---|---|---|---|
| 1 | T | T | T | F | F | T | F |
| 2 | T | F | F | F | F | T | T |
| 3 | F | T | T | T | T | F | F |
| 4 | F | F | T | T | T | T | T |
|  | (i) | (ii) | (iii) | (iv) | (v) | (vi) | (vii) |

The truth-value "F" under the " ⊃ " with the largest scope (vi) in the above truth table (row 3) show that the truth function is not a tautology since in some cases it is true (rows 1, 2, and 4) and in one case false (row 3). Therefore, the truth function is a contingency. The implicative argument which is correlated with the contingent truth function is invalid as are all arguments having the same logical form.

The two preceding arguments are examples of the implicative type of argument mentioned in the introductory section. Keep in mind that within each row of a truth table, the truth-value of a particular variable remains constant. Further, as mentioned before, the expression for which a particular variable has been substituted remains constant throughout all rows of a truth table for a given argument.

Previously, implicative arguments were distinguished from one other type of argument – the

declarative type. The following is a truth-functional formulation of a contradictory declarative type; hence, neither the label "valid" nor the label "invalid" applies. However, for every contradictory declarative argument, there is an invalid implicative argument.

(p · ~p)

| p | ~p | (p · ~ p) |
|---|-----|-----------|
| T | F | F |
| F | T | F |

The negation of a contradictory truth function yields a tautologous truth function; negating a tautologous truth function yields a contradictory truth function. Valid arguments expressed (that is, formulated) as implicative truth functions yield tautologies. Invalid arguments yield either contingent or contradictory truth functions, never tautologies.

## E. SHORTHAND TECHNIQUE TESTING VALIDITY OF ARGUMENTS

The truth table method for testing the validity of arguments is admittedly a mechanical method. Its construction and application become cumbersome with extended arguments. For example, an argument with six different sentential variables requires $2^6$ or 64 rows in the truth table.

Fortunately, there is a shorthand technique for testing the validity of an argument. This method attempts to prove invalidity. Consider the following argument form:

p ⊃ q

q

____

∴ p

If it is possible to assign truth values to the variables in the above argument form such that the premises are true and the conclusion false, then the argument is invalid; if such assignment cannot be made, then the argument form is valid. If the argument form is valid and the conclusion false, then at least one of the premises will be, must be, false.

Step 1: Assign the truth-value "F" to the conclusion variable.

p ⊃ q

q

____

∴ p    **F**

Step 2: Assign the truth-value "T" to the premise variable "q". (Assigning the truth-value "F" to "q" at this point defeats the shorthand technique, since the application of this method is an attempt to

prove invalidity.)

p ⊃ q

q      **T**

_____

∴ p    **F**

Step 3: Carry out the same assignment of truth values to the variables in the premise " p ⊃ q ".

F    T

p ⊃ q  **T**

q        **T**

_____

∴ p    **F**

It has been shown above that it is possible to assign truth values to the variables "p" and "q" of the argument form such that the premises are true and the conclusion false. (One row in a truth-table analysis would show and "F" under the " ⊃ " with the largest scope). Therefore, the argument is not valid; if not valid, then invalid. This form of invalid argument has been traditionally called the fallacy of affirming the consequent.

Consider the following argument form (affirming the antecedent):

T    F

p ⊃ q  **F**

p        **T**

_____

∴ q    **F**

It is not possible to assign truth values to the conclusion and premise variables of the above argument form such that the premises are true and the conclusion false. Therefore, the argument is not invalid. If not invalid, then valid. This valid argument form has been labeled modus ponens.

One other well-known fallacy is known as the fallacy of *denying the antecedent*. Consider, for example, the following:

F    F

p ⊃ q  **T**

**F** (p)

~ p    **T**

_____

∴ q   **F**

Note however, denying the consequent variable "q," to detach the denial of the antecedent variable "p" of "p ⊃ q" as the conclusion variable "~ p" is not a fallacy, as shown in what follows..

T    F

p ⊃ q  **F**

F

~ q    **T**

_____

   T

∴ ~p  **F**

This valid argument form is recognized as modus tollens.

In the next section, we shall give a list of some valid argument forms (sometimes called valid-inference rules). It should be obvious that a valid argument form formulated as a truth function will always yield a tautology.

In sum, the shorthand technique represents a relatively easy method of checking for validity. By use of this technique cumbersome construction of full truth tables can be avoided.

In connection with the topics of validity, truth-value, and material implication, a note on a common bit of terminology is in order. For a valid argument, it is *sufficient* that the antecedent be true for the consequent to be materially implied as true. Moreover, for a valid argument it is *necessary* that the consequent be true for that consequent to be materially implied by a true antecedent. Accordingly, the antecedent of the material-implication relation is often referred to as the sufficient condition. The consequent is then referred to as the necessary condition. The labels "sufficient" and "necessary" are undoubtedly used in other senses, but it is enough that the student of elementary logic to understand how the two words are used in talking about logical operations as exhibited herein.

## F. VALID-INFERENCE RULES

1. Detachment Rules

Elementary Detachment Rules

| 1. | (p ⊃ q) , p | ∴ q | modus ponens (MP) |
|---|---|---|---|
| 2. | p , q | ∴ (p · q) | conjunction (C) |
| 3. | (p · q) | ∴ p | simplification (S) |
| 4. | p | ∴ ( p v q) | addition (A) |
| 5. | (p ⊃ q) , (q ⊃ r) | ∴ p ⊃ r | hypothetical syllogism (HS) |

Derived Detachment Rules

| 6. | (p ⊃ q) , ~ q | ∴ ~ p | modus tollens (MT) |
|---|---|---|---|
| 7. | (p v q) , ~ p | ∴ q | disjunctive syllogism (DS) |
| 8. | (p ⊃ q) · (r ⊃ s) , (p v r) | ∴ (q v s) | constructive dilemma (CD) |
| 9 | (p ⊃ q) · (r ⊃ s), (~q v ~s) | ∴ (~ p v ~r ) | destructive dilemma (DD) |

2. Replacement Rules

Elementary Replacement Rules

| 10. | (p ⊃ q) ≡ (~ p v q ) | material implication (MI) |
|---|---|---|
| 11. | (p ≡ q) ≡ [( p · q ) v ( ~p · ~q)]<br><br>(p ≡ q) ≡ [( p ⊃·q ) · (q ⊃ p)] | material equivalence (ME) |
| 12. | p ≡ (p · p)<br><br>p ≡ (p v p) | tautology (T) |
| 13. | (p · q ) ≡ (q · p )<br><br>(p v q ) ≡ (q v p ) | commutation (CN) |
| 14. | p ≡ ~ ~ p | double negation (DN) |
| 15. | [(p · (q · r)] ≡ [(p · q) · r ]<br><br>[(p v (q v r)] ≡ [(p v q) v r ] | association (AN) |
| 16 | [(p v ( q · r)] ≡ [( p v q) · ( p v r )]<br><br>[(p · ( q v·r)] ≡ [( p · q) v ( p · r )] | distribution (DTN) |

Derived Replacement Rules

| 17. | [(p · q) ⊃ r] ≡ [p ⊃ (q ⊃ r )] | exportation (EN) |
|---|---|---|
| 18. | ~ (p · q ) ≡ ( ~p v ~ q)<br><br>~ (p v q ) ≡ ( ~p · ~ q) | De Morgan's theorem (DeM) |

| 19. | $(p \supset q) \equiv (\sim q \supset \sim p)$ | transposition (TN) |
|---|---|---|

Rules 1 through 9 above, the detachment rules are classified into elementary rules (1 through 5) and derived rules (6 through 9). These detachment rules apply solely to whole lines in a formal proof, as will be shown in the section dealing with formal proofs.

The replacement rules (10 through 19) apply to whole lines or to portions of whole lines in a formal proof. Rules 10 through 16 are elementary; rules 17 through 19 are derived. These replacement rules allow for substitution of logically equivalent formulations wherever they occur in a formal proof.

# G. FORMAL PROOFS

The truth-table method and the shorthand technique are decision procedures that can be applied to truth-functional formulations to determine whether or not any given truth function is a tautology. There is another method by which it can be demonstrated that a given conclusion follows from certain premises. This is the method of deriving a conclusion from premises by making use of valid-inference rules. This derivation, however, is not a decision procedure. It cannot tell us whether any given implicative argument is valid. It does, however, enable us to show how a certain truth-functional expression (i.e., the conclusion of an argument) is derivable from another truth-functional expression (i.e., the premise of that argument) just in case the two truth-functional expressions are tautologously related (or, the argument valid). As such, formal proof procedures provide a useful method for exhibiting the validity status of arguments.

In constructing a formal proof, each step in the derivation is justified in terms of a valid inference rule. For example, consider the following argument:

1. If Leo is smart, he will not prosecute Andy.

2. If he does not prosecute Andy, then Judy will lose face.

3. If Judy loses face, then Nancy will be unhappy.

4. Therefore, if Leo is smart, Nancy will be unhappy.

Now, let p = Leo is smart; q = he will prosecute Andy; r = Judy will lose face; s = Nancy will be unhappy. Truth-functional formulation of the above then results in the following argument form:

1. $p \supset \sim q$

2. $\sim q \supset r$

3. $r \supset s$

4. $\therefore p \supset s$

It is worth noting that every valid argument has a proof whether we manage to discover a proof, or not; however, there is no proof for an invalid argument. Having found that the above argument form is valid, we proceed to derive the conclusion from the given premises by constructing a formal proof showing a step-by-step justification using valid-inference rules to justify each step in the following

way:

1. $p \supset \sim q$

2. $\sim q \supset r$

3. $r \supset s$       $\angle \therefore p \supset s$

4. $p \supset r$       (from 1, 2 by HS)

5. $p \supset s$       (from 4, 3 by HS)

The symbol "$\angle$" introduces that which is to be proved. As a matter of convention, it is placed to the right of the last premise given.

Consider another argument form:

1. $\sim p \vee q$

2. $\sim (q \cdot \sim r)$

3. $r \supset s$

4. $\sim s$

5. $\therefore \sim p$

Again, the argument form is valid and we can construct a formal proof showing a step-by-step justification in terms of valid-inference rules.

1. $\sim p \vee q$

2. $\sim (q \cdot \sim r)$

3. $r \supset s$

4. $\sim s$       $\angle \therefore \sim p$

5. $p \supset q$       (from 1 by MI)

6. $\sim q \vee \sim \sim r$       (from 2 by DeM)

7. $\sim q \vee r$       (from 6 by DN)

8. $q \supset r$       (from 7 by MI)

9. $p \supset r$       (from 5, 8 by HS)

10. $p \supset s$       (from 9, 3 by HS)

11. $\sim p$       (from 10, 4 by MT)

We have so far examined one method for ascertaining the validity of arguments, namely, the truth-table method discussed in Section D, and one method of demonstrating the validity of arguments,

namely, the formal-proof method discussed above. Just as the shorthand technique (Section E) represents an economy upon the full truth-table method, two variants of the formal proof method frequently (although not always) represent an economy in demonstrative proof. These are called conditional proof and indirect proof. Since the label *reductio ad absurdum* is applied sometimes to the shorthand truth-table technique and sometimes to the indirect method of proof, it is in the interest of clarity to distinguish the two senses.

# H. THE CONDITIONAL-PROOF METHOD

The conditional-proof method and the formal-proof method are similar in that both require step-by-step justification. The conditional-proof method, while easier to use in many cases, is restricted to arguments with implicative conclusions and their logical equivalents.

For example, if we let "B" stand for a premise set, the conditional-proof method can be used whenever the conclusion is of the following logical form: "p ⊃ q", or the logically equivalent "~ p v q" or "~(p · ~q)". When conducting conditional proof the conclusion to be proved should be formulated in the " ⊃ " notation prior to making the conditional assumption step.

If "B" is the premise set, and "p ⊃ q" is the conclusion, we can formulate this argument in the following way:

B ⊃ (p ⊃ q)

But "B ⊃ (p⊃ q)" is logically equivalent to "(B · p) ⊃ q" as a truth table analysis will show. This equivalence means that if both "B" and "p" conjoined imply "q", then "B" alone implies "p ⊃ q." This logical equivalence is known as *exportation*, shown in the list under valid inference rules.

In all such arguments, then, the antecedent in the implicative conclusion, "p" in this case, is assumed as part of the premise set. We can prove that "q," the consequent in the implicative conclusion, follows from the premise set and " p " conjoined.

Consider this argument form:

l. (p ⊃ q) ⊃ s

_____

∴ ~ s ⊃ ~ q

Constructing a formal proof results in the following derivation:

1. (p ⊃ q) ⊃ s          ∠ ∴~ s ⊃ ~q

2. ~s ⊃ ~ (p ⊃ q)          (1, TN)

3. ~s ⊃ ~ ( ~ p v q)          (2, MI)

4 ~s ⊃ (~ ~ p · ~ q)          (3, DeM)

5. ~s ⊃ (p · ~ q)          (4, DN)

6. s v (p · ~ q)                    (5, MI)

7. (s v p) · (s v· ~q)              (6, DTN)

8. (s v ~q) · (s v p)               (7, CN)

9. s v ~ q                          (8, S)

10. ~ s ⊃ ~ q                       (9, MI)

A conditional proof proceeds in the same fashion as the proof above. There are, however, some differences. Indentation, for example, of steps 2 through 7 in the following proof indicates the steps in which the conditional proof is used. Lines 1 and 8 are not indented since these lines are not within the scope of the assumption introduced in line 2. Indentation within indentation is possible whenever the conclusion has an implicative as the antecedent or consequent of the main implicative. In such cases, one assumption is introduced within the other assumption in the conditional proof. In what follows, we are considering only one such assumption.

1. (p ⊃ q) ⊃ s              ∠ ∴ ~ s ⊃ ~ q

    2. ~ s                  Assump. ∠ ∴ ~ q

    3 ~ (p ⊃ q)             (1, 2 MT)

    4. ~ (~p v q)           (3, MI)

    5. ~~p · ~q             (4, DeM)

    6. ~q · ~ ~p            (5, CN)

    7. ~q                   (6, S)

8. ~ s ⊃ ~ q                (2 through 7 by CP)

It should be obvious that the formal proof is longer and perhaps more difficult to construct than the conditional proof.

# I. THE INDIRECT-PROOF METHOD

By application of the inference rules for addition, material implication, and exportation in that order to the conclusion of any argument, a warrant is provided for including in the premise set the negation of the initial conclusion. This operation in simplest form can be shown as follows:

Given: "p ∴ q". Then such by Addition yields "p ∴ (q v q)".

Given: "p ∴ (q v q)". Then such by Material Implication of "(q v q)" yields "p ∴ (~ q ⊃ q)".

Given: "p ∴ (~ q ⊃ q)". Then such yields "(p · ~q) ∴ q" by Exportation.

In the proof procedure the additional premise is used to derive contradictory premises. Next, the conclusion to be proved is added to the positive member of the contradictory pair of premises. Then

the result of the second operation is conjoined with the negative member of the contradictory pair to yield by disjunctive syllogism the conclusion to be proved. The indentation device used in conditional proof is adopted for indirect proof as well.

Consider the following example:

1. p v (q ⊃ r)

2. q · ~ r                    ∠ ∴ p

        3. ~ p              Assuming. ∠ ∴ p

        4. q ⊃ r           (1, 3 DS)

        5. q               (2, S)

        6. r               (4, 5 MP)

        7. ~r · q          (2, CN)

        8. ~ r             (7, S)

        9. r v p           (6, A)

10. p                       (9, 8 DS)

Note that lines 6 and 8 form a contradictory pair (r · ~ r); at line 9 the conclusion is added to the positive member of that pair; then by application of DS to lines 9 and 8 the conclusion is derived. Note. The indirect method may be used within a conditional proof (CP) as exemplified by the following:

1. (p · q) ⊃ r

2. (q ⊃ r) ⊃ s                        ∠ ∴ p ⊃ s

        3. p                          Assuming ∠ ∴ s

                4. ~ s               Assuming ∠ ∴ s

                5. p ⊃ (q ⊃ r)       (1 EN)

                6. q ⊃ r             (5, 3 MP)

                7. ~ (q ⊃ r)         (2, 4 MT)

                8. (q ⊃ r) v s       (6, A)

        9. s                          (8, 7 DS)

10. p ⊃ s                            (3 through 9, CP)

Note that in such case indentation for indirect-proof steps occurs within the indentation for conditional-proof steps. Lines 6 and 7 form a contradictory pair: (q ⊃ r) and ~ (q ⊃ r). Further, this

proof illustrates the point that conditional-proof or indirect-proof method does not always serve economy, for the proof of this argument conducted by formal method is shorter than that provided by either conditional or indirect method.

Witness the following:

1. $(p \cdot q) \supset r$

2. $(q \supset r) \supset s$        $\angle \therefore \ (p \supset s)$

3. $p \supset (q \supset r)$        (1, EN)

4. $p \supset s$        (3, 2 HS)

Note: In modern as in classical logic, quantitative and qualitative (in the sense of affirmation and negation) considerations must not be neglected. The procedures for handling these distinctions in connection with the symbolic techniques presented are material for additional material.

## SUMMARY

According to Gordon Clark, in a note of In Defense of Theology (pp. 49-51) observes that contemporary logicians reject A(ab) E(cb) < O(ca) as invalid, reducing the number of valid syllogisms from 24 to 15. He explains first how this rejection came about, then why it is not necessary. Although the argument for the reduction of syllogisms has no logical flaw, Clark finds it is based on a "great blunder."

> *"I now assert that Russell made a great blunder, not in his deductions, but in his definition of the term All. Remember he explained All a is b as a is included in b. However obvious Russell's definition seems, it is not a correct analysis of the English word All. … Contemporary logic is based on a misunderstanding of the English word all." (Clark, Reference 1, p. 51)*

Clark, in Logic (p. 81f), provides a definition of *all* that, though awkward and not free from all difficulties, is in accord with ordinary English's *all*. It preserves subalternation and restores valid syllogisms to 24.

"Russell's definition of All is faulty, and his completely valid deductions from this faulty definition have nothing to do with a*ll*, s*ome*, or subalternation." (Clark, Reference 2. p. 83)

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References

*For a critique of "material implication" see Gordon H. Clark, Logic, The Trinity Foundation, Unicoi, TN, pp. 35-38. His definition of a valid argument is adequate for all versions of implicatives "An inference is valid if the form of the conclusion is true every time the forms of the premise are."

1. Clark, Gordon H. In Defense of Theology. Unicoi, TN: The Trinity Foundation. 2007.

2. _____. Logic. 3rd ed. Unicoi, TN: The Trinity Foundation. 1998.

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